TUTORIAL NOTES FOR MATH4220

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1. d'Alembert formula and wave equations

Let us recall the d'Alembert formula for the one dimensional Cauchy problems of wave equations.

Theorem 1 (d'Alembert formula)**.** *Suppose* $u_0 \in C^2(\mathbb{R})$*,* $u_1 \in C^1(\mathbb{R})$*,* $f \in C^1(\mathbb{R})$*. Then*

$$\begin{aligned} \partial_t^2 u - \partial_x^2 u = f, &\quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ (u, \partial_t u)(0,x) = (u_0, u_1)(x), &\quad x \in \mathbb{R}, \end{aligned}$$

has a solution $u \in C^2(\mathbb{R}_+ \times \mathbb{R})$ *which is defined by*

$$u(t,x) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)dyds.$$

In the following, we discuss some applications of d'Alembert formula to study the half space problems for wave equations.

Example 2. Suppose $u_0 \in C^2(\mathbb{R}_+)$, $u_1 \in C^1(\mathbb{R}_+)$, $u_b \in C^2(\mathbb{R}_+)$, $f \in C^1(\mathbb{R}_+)$ and the compatibility condition holds

$$u_0(0) = u_b(0), \quad u_1(0) = u_b'(0), \quad u_0''(0) = u_b''(0).$$

. Then

$$\begin{aligned} \partial_t^2 u - \partial_x^2 u = f, &\quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ (u, \partial_t u)(0,x) = (u_0, u_1)(x), &\quad x \in \mathbb{R}_+, \\ u(t,0) = u_b(t), &\quad t \in \mathbb{R}_+, \end{aligned}$$

has a solution $u \in C^2(\mathbb{R}_+ \times \mathbb{R}_+)$. Moreover, for $x \geq t$,

$$u(t,x) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)dyds,$$

for $0 \leq x < t$,

$$\begin{aligned} u(t,x) =& u_b(t-x) + \frac{1}{2}(u_0(x+t) - u_0(-x+t)) + \frac{1}{2}\int_{-x+t}^{x+t} u_1(y)dy \\ &+ \frac{1}{2}\int_0^{t-x} \int_{-x+(t-s)}^{x+(t-s)} f(s,y)dyds + \frac{1}{2}\int_{t-x}^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)dyds. \end{aligned}$$

Solution. Denote

$$v(t,x) = u(t,x) - u_b(t),$$

then

$$\begin{aligned} \partial_t^2 v - \partial_x^2 v = g, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ (v, \partial_t v)(0,x) = (v_0, v_1)(x), \quad x \in \mathbb{R}_+, \\ v(t,0) = 0, \quad t \in \mathbb{R}_+, \end{aligned}$$

where $g(t,x) := f(t,x) - \partial_t^2 u_b(t)$, $v_0(x) = u_0(x) - u_b(0)$, $v_1(x) = u_1(x) - \partial_t u_b(0)$. To solve the above problem, we set

$$\tilde{g}(t,x) = \begin{cases} g(t,x), & x \geq 0, \\ -g(t,-x), & x < 0. \end{cases}$$

$$\tilde{v_0}(x) = \begin{cases} v_0(x), & x \geq 0, \\ -v_0(-x), & x < 0. \end{cases}$$

$$\tilde{v_1}(x) = \begin{cases} v_1(x), & x \geq 0, \\ -v_1(-x), & x < 0. \end{cases}$$

Let $\tilde{v}$ be the solution to the following problem

$$\begin{aligned} \partial_t^2 \tilde{v} - \partial_x^2 \tilde{v} = \tilde{g}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ (\tilde{v}, \partial_t \tilde{v})(0,x) = (\tilde{v}_0, \tilde{v_1})(x), \quad x \in \mathbb{R}. \end{aligned}$$

We claim that $v = \tilde{v}$ when we restrict x to $\mathbb{R}_+$. Then it suffices to prove $\tilde{v}(t,0) = 0$ for all $t \in \mathbb{R}_+$. Indeed, by the d'Alembert formula,

$$\tilde{v}(t,x) = \frac{1}{2}(\tilde{v}_0(x+t) + \tilde{v}_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{v}_1(y)dy + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} \tilde{g}(s,y)dyds,$$

then it can be checked that $\tilde{v}(t,0) = 0$. Therefore for $x \geq t$,

$$u(t,x) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)dyds,$$

for $0 \leq x < t$,

$$\begin{aligned} u(t,x) =& u_b(t) + \frac{1}{2}(v_0(x+t) - v_0(-x+t)) \\ &+ \frac{1}{2}\int_{x-t}^{0} -v_1(-y)dy + \frac{1}{2}\int_0^{x+t} v_1(y)dy \\ &+ \frac{1}{2}\int_0^{t-x} \int_{x-(t-s)}^{0} -g(s,-y)dyds \\ &+ \frac{1}{2}\int_0^{t-x} \int_0^{x+(t-s)} g(s,y)dyds + \frac{1}{2}\int_{t-x}^{t} \int_{x-(t-s)}^{x+(t-s)} g(s,y)dyds \\ =& u_b(t-x) + \frac{1}{2}(u_0(x+t) - u_0(-x+t)) + \frac{1}{2}\int_{-x+t}^{x+t} u_1(y)dy \\ &+ \frac{1}{2}\int_0^{t-x} \int_{-x+(t-s)}^{x+(t-s)} f(s,y)dyds + \frac{1}{2}\int_{t-x}^{t} \int_{x-(t-s)}^{x+(t-s)} f(s,y)dyds. \end{aligned}$$

Example 3. Suppose $u_0 \in C^2(\mathbb{R}_+)$, $u_1 \in C^1(\mathbb{R}_+)$, $u_b \in C^1(\mathbb{R}_+)$, $f \in C^1(\mathbb{R}_+)$ and the compatibility condition holds

$$u'(0) = u_b(0), \quad u_1'(0) = u_b'(0).$$

Then

$$\begin{aligned} \partial_t^2 u - \partial_x^2 u = f, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ (u, \partial_t u)(0,x) = (u_0, u_1)(x), \quad x \in \mathbb{R}_+, \\ \partial_x u(t,0) = u_b(t), \quad t \in \mathbb{R}_+, \end{aligned}$$

has a solution $u \in C^2(\mathbb{R}_+ \times \mathbb{R}_+)$. Moreover, for $x \geq t$,

$$u(t,x) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)dyds,$$

for $0 \leq x < t$,

$$\begin{aligned} u(t,x) = &- \int_0^{t-x} u_b(s)ds + \frac{1}{2}(u_0(x+t) + u_0(-x+t)) \\ &+ \frac{1}{2}\int_0^{t-x} u_1(y)dy + \frac{1}{2}\int_0^{x+t} u_1(y)dy \\ &+ \frac{1}{2}\int_0^{t-x}\int_0^{-x+(t-s)} f(s,y)dyds + \frac{1}{2}\int_0^{t-x}\int_0^{x+(t-s)} f(s,y)dyds \\ &+ \frac{1}{2}\int_{t-x}^{t}\int_{x-(t-s)}^{x+(t-s)} f(s,y)dyds. \end{aligned}$$

Solution. Denote

$$v(t,x) = u(t,x) - xu_b(t),$$

then

$$\begin{aligned} \partial_t^2 v - \partial_x^2 v = g, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ (v, \partial_t v)(0,x) = (v_0, v_1)(x), \quad x \in \mathbb{R}_+, \\ \partial_x v(t,0) = 0, \quad t \in \mathbb{R}_+, \end{aligned}$$

where $g(t,x) := f(t,x) - x\partial_t^2 u_b(t)$, $v_0(x) = u_0(x) - xu_b(0)$, $v_1(x) = u_1(x) - x\partial_t u_b(0)$. To solve the above problem, we set

$$\tilde{g}(t,x) = \begin{cases} g(t,x), & x \geq 0, \\ g(t,-x), & x < 0. \end{cases}$$

$$\tilde{v_0}(x) = \begin{cases} v_0(x), & x \geq 0, \\ v_0(-x), & x < 0. \end{cases}$$

$$\tilde{v_1}(x) = \begin{cases} v_1(x), & x \geq 0, \\ v_1(-x), & x < 0. \end{cases}$$

Let $\tilde{v}$ be the solution to the following problem

$$\begin{aligned} \partial_t^2 \tilde{v} - \partial_x^2 \tilde{v} = \tilde{g}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ (\tilde{v}, \partial_t \tilde{v})(0,x) = (\tilde{v_0}, \tilde{v_1})(x), \quad x \in \mathbb{R}. \end{aligned}$$

We claim that $v = \tilde{v}$ when we restrict x to $\mathbb{R}_+$. Then it suffices to prove $\partial_x \tilde{v}(t,0) = 0$ for all $t \in \mathbb{R}_+$. Indeed, by the d'Alembert formula,

$$\tilde{v}(t,x) = \frac{1}{2}(\tilde{v_0}(x+t) + \tilde{v_0}(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{v_1}(y)dy + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} \tilde{g}(s,y)dyds,$$

then it can be checked that $\partial_x \tilde{v}(t,0) = 0$. Therefore for $x \geq t$,

$$u(t,x) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)dyds,$$

for $0 \leq x < t$,

$$\begin{aligned}
u(t,x) =& u_b(t) + \frac{1}{2}(v_0(x+t) - v_0(-x+t)) \\
&+ \frac{1}{2}\int_{x-t}^{0} v_1(-y)dy + \frac{1}{2}\int_0^{x+t} v_1(y)dy \\
&+ \frac{1}{2}\int_0^{t-x}\int_{x-(t-s)}^{0} g(s,-y)dyds \\
&+ \frac{1}{2}\int_0^{t-x}\int_0^{x+(t-s)} g(s,y)dyds + \frac{1}{2}\int_{t-x}^{t}\int_{x-(t-s)}^{x+(t-s)} g(s,y)dyds \\
=& -\int_0^{t-x} u_b(s)ds + \frac{1}{2}(u_0(x+t) + u_0(-x+t)) \\
&+ \frac{1}{2}\int_0^{t-x} u_1(y)dy + \frac{1}{2}\int_0^{x+t} u_1(y)dy \\
&+ \frac{1}{2}\int_0^{t-x}\int_0^{-x+(t-s)} f(s,y)dyds + \frac{1}{2}\int_0^{t-x}\int_0^{x+(t-s)} f(s,y)dyds \\
&+ \frac{1}{2}\int_{t-x}^{t}\int_{x-(t-s)}^{x+(t-s)} f(s,y)dyds.
\end{aligned}$$

A Supplementary Problem

Problem 4. Show that the solution u to one dimensional wave equation,

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

satisfies that for arbitrary $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and $\xi, \tau \in \mathbb{R}$,

$$u(t,x) + u(t+\xi+\tau, x+\xi-\tau) = u(t+\xi, x+\xi) + u(t+\tau, x-\tau).$$

For more materials, please refer to [1, 2, 3, 4].

References

[1] S. Alinhac, *Hyperbolic partial differential equations*, Universitext, Springer, Dordrecht, 2009.

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